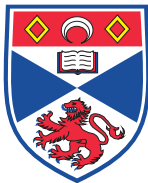


# Finitely Presented Semigroups

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## Definition

A **semigroup** is a set  $S$  together with a binary operation  $*$  :  $S \times S \rightarrow S$  such that

$$(x * y) * z = x * (y * z)$$

for all  $x, y, z \in S$ .

- We may write  $xy$  instead of  $x * y$
- Can we describe all semigroups in the same way?

# Free Semigroups

- Let  $X$  be an alphabet, e.g.  $\{a, b, c\}$

## Definition

A **word** over  $X$  is a finite ordered list of letters from  $X$ .

e.g.  $abaacbba$

## Definition

The **free semigroup**  $X^+$  is the set of all words over  $X$  with the operation of concatenation.

e.g.  $aba * cab = abacab$

- Concatenation is associative:  $(w_1w_2)w_3 = w_1(w_2w_3)$
- $X^+$  is infinite
- If  $|X| = |Y|$  then  $X^+ \cong Y^+$

- We can create other semigroups from free semigroups
- Consider  $X = \{a, b\}$
- $X^+ = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, \dots\}$
- We can **identify** two elements and take a quotient

## Example

Let  $S$  be a semigroup where  $ab = ba$ .

Now

$$a\underline{ab} = \underline{aba},$$

$$a\underline{aba} = \underline{abaa},$$

$$ab\underline{baa} = \underline{ababaa},$$

and so on.

In  $S$ , we can commute  $a$  and  $b$  however we like.

# Semigroup Presentations

We can write  $S$  using a presentation

## Definition

If  $X$  is an alphabet and  $R$  a set of relators (pairs of words over  $X$ ) then

$$\langle X | R \rangle$$

is a **presentation** for the semigroup defined by taking the free semigroup  $X^+$  and identifying each pair in  $R$ .

## Example

In our last example,  $X = \{a, b\}$  and  $R = \{(ab, ba)\}$ .

Our semigroup  $S$  has a presentation

$$\langle a, b \mid ab = ba \rangle.$$

# Normal Form

- In a finitely presented semigroup, one element may be represented by many different strings
- A **normal form** for  $S$  is a set of words such that each element of  $S$  appears *precisely once*

## Example

In our running example  $S = \langle a, b \mid ab = ba \rangle$ , elements commute however we want. Move  $as$  left and  $bs$  right as much as we can:

$$abba = ab\underline{ab} = a\underline{abb}, \quad abbaaaba = aaaaabbb = a^5b^3.$$

This gives us the normal form  $\{a^i b^j : i, j \in \mathbb{N}\}$ .  
It turns out  $S$  is isomorphic to the direct product  $\mathbb{N} \times \mathbb{N}$ .

## Other free objects

Special categories of semigroups have their own free objects.

### Example

A **free monoid**  $X^*$  is the free semigroup  $X^+$  with an appended identity, the empty word  $\varepsilon$ .

### Example

A **free group**  $F_X$  has an identity, and uses the alphabet  $X \cup X^{-1}$ , where each letter  $a$  has an inverse  $a^{-1}$  such that  $aa^{-1} = a^{-1}a = \varepsilon$ .

### Example

A **free abelian group** adds relators to  $F_X$  so that all letters commute.

### Example

A **free band** adds relators to  $X^+$  so that  $ww = w$  for any word  $w$ . It turns out to be finite!

Thank you