Finitely Presented Semigroups

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Semigroups

Definition

A **semigroup** is a set S together with a binary operation $*: S \times S \to S$ such that

$$(x * y) * z \quad = \quad x * (y * z)$$

for all $x, y, z \in S$.

- We may write xy instead of $x * y$
- Can we describe all semigroups in the same way?

Free Semigroups

- Let X be an alphabet, e.g. $\{a, b, c\}$

Definition

A **word** over X is a finite ordered list of letters from X.
e.g. $abaacbba$

Definition

The **free semigroup** X^+ is the set of all words over X with the operation of concatenation.
e.g. $aba * cab = abacab$

- Concatenation is associative: $(w_1 w_2) w_3 = w_1 (w_2 w_3)$
- X^+ is infinite
- If $|X| = |Y|$ then $X^+ \cong Y^+$

Relators

- We can create other semigroups from free semigroups
- Consider $X = \{a, b\}$
- $X^+ = \{a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, \ldots\}$
- We can **identify** two elements and take a quotient

Example

Let S be a semigroup where $ab = ba$.
Now

$$a\underline{ab} = a\underline{ba},$$

$$a\underline{ab}a = a\underline{ba}a,$$

$$ab\underline{ba}a\underline{ab}a = ab\underline{ab}a\underline{ba}a,$$

and so on.
In S, we can commute a and b however we like.

Semigroup Presentations

We can write S using a presentation

Definition

If X is an alphabet and R a set of relators (pairs of words over X) then

$$\langle X|R\rangle$$

is a **presentation** for the semigroup defined by taking the free semigroup X^+ and identifying each pair in R.

Example

In our last example, $X = \{a, b\}$ and $R = \{(ab, ba)\}$.
Our semigroup S has a presentation

$$\langle a, b \mid ab = ba\rangle .$$

Normal Form

- In a finitely presented semigroup, one element may be represented by many different strings
- A **normal form** for S is a set of words such that each element of S appears *precisely once*

Example

In our running example $S = \langle a, b \mid ab = ba \rangle$, elements commute however we want. Move as left and bs right as much as we can:

$$abba = ab\underline{ab} = a\underline{ab}b, \quad abbaaaba = aaaaabbb = a^5b^3.$$

This gives us the normal form $\{a^i b^j : i, j \in \mathbb{N}\}$.
It turns out S is isomorphic to the direct product $\mathbb{N} \times \mathbb{N}$.

Other free objects

Special categories of semigroups have their own free objects.

Example

A **free monoid** X^* is the free semigroup X^+ with an appended identity, the empty word ε.

Example

A **free group** F_X has an identity, and uses the alphabet $X \cup X^{-1}$, where each letter a has an inverse a^{-1} such that $aa^{-1} = a^{-1}a = \varepsilon$.

Example

A **free abelian group** adds relators to F_X so that all letters commute.

Example

A **free band** adds relators to X^+ so that $ww = w$ for any word w. It turns out to be finite!

Thank you