Congruences of the Partition Monoid

Based on joint work with J. East, J.D. Mitchell, and N. Ruškuc

Michael Torpey

University of St Andrews

2017-05-17

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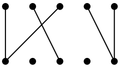
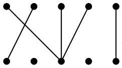
$\alpha =$ , $\beta =$ ,

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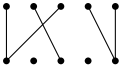
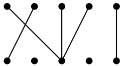
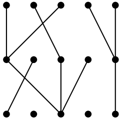
$\alpha =$  , $\beta =$  , $\alpha\beta =$ 

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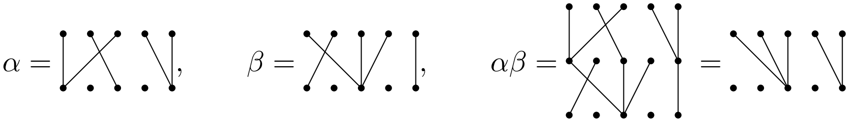

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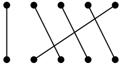
$\alpha =$  ,

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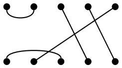
$\alpha =$  ,

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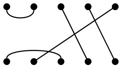
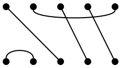
$\alpha =$  , $\beta =$ 

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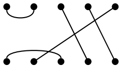
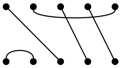
$\alpha =$  , $\quad \beta =$ 

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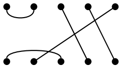
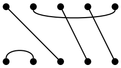
$\alpha =$  , $\beta =$ 

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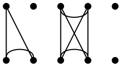
$\gamma =$  ,

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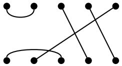
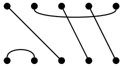
$\alpha =$  , $\beta =$ 

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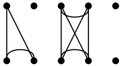
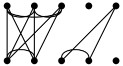
$\gamma =$  , $\delta =$ 

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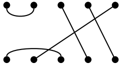
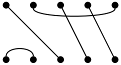
$\alpha =$  , $\beta =$ 

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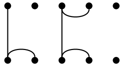
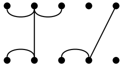
$\gamma =$  , $\delta =$ 

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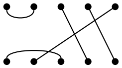
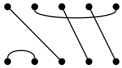
$\alpha =$  , $\beta =$ 

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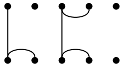
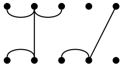
$\gamma =$  , $\delta =$ 

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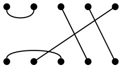
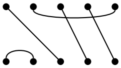
$\alpha =$  , $\beta =$ 

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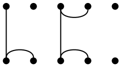
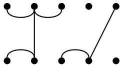
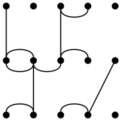
$\gamma =$  , $\delta =$ 

$\gamma\delta =$ 

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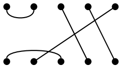
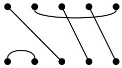
$\alpha =$  , $\quad \beta =$ 

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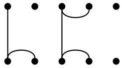
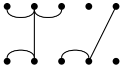
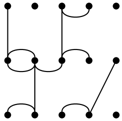
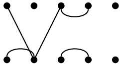
$\gamma =$  , $\quad \delta =$ 

$\gamma\delta =$  $=$ 

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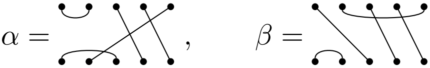

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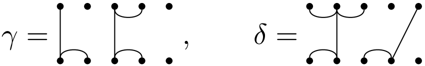

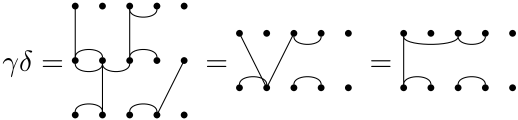

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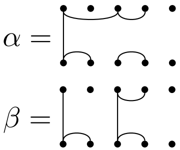

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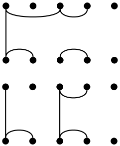

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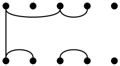
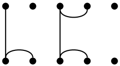
$\alpha =$ 

$\beta =$ 

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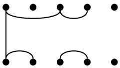
$\alpha =$ 

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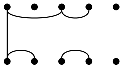
$\alpha =$ 

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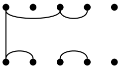
$\alpha =$ 

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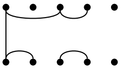
$\alpha =$ 

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Attributes of partitions

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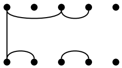
The **domain** (*resp.* **codomain**) of a partition $\alpha \in \mathcal{P}_n$ is the set of points $i \in \mathbf{n}$ (*resp.* $i' \in \mathbf{n}'$) which lie in transversal blocks.

Definition

The **kernel** (*resp.* **cokernel**) of a partition $\alpha \in \mathcal{P}_n$ is the restriction of α to $\mathbf{n}$ (*resp.* $\mathbf{n}'$).

$\alpha =$ 

$$\operatorname{dom}\alpha = \{1, 3, 4\}, \quad \operatorname{codom}\alpha = \{1', 2'\},$$

$$\ker\alpha = \{\{1, 3, 4\}, \{2\}, \{5\}\}, \quad \operatorname{coker}\alpha = \{\{1', 2'\}, \{3', 4'\}, \{5'\}\}$$

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[2]

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Example

If a semigroup S has an ideal I, then the *Rees congruence* $\rho_I = \Delta_S \cup (I \times I)$ is a congruence on S.

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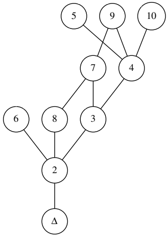

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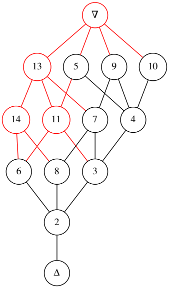

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All Rees congruences of $\mathcal{M}_n$ have the form

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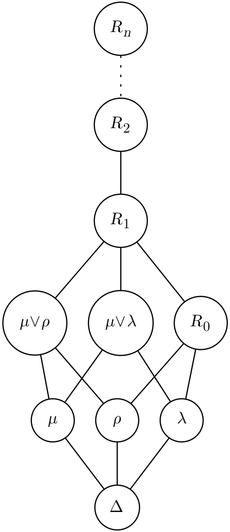

Thank you for listening

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